

Line-Integral Representations of the Diffraction of Scalar Fields

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Abstract

Traditionally, the diffraction of a scalar wave satisfying Helmholtz equation through an aperture on an otherwise black screen can be solved approximately by Kirchhoff's integral over the aperture. Rubinowicz, on the other hand, was able to split the solution into two parts: one is the geometrical part that appears only in the geometrical illuminated region, and the other representing the reflected wave is a line-integral along the edge of the aperture. However, this decomposition is not entirely satisfactory in the sense that the two separated fields are discontinuous at the boundary of the illuminated region. Also, the functional form of the line-integral is not what one would expect an ordinary reflection wave should be due to some confusing factors in the integrand. Finally, the boundary conditions on the screen imposed by Kirchhoff's approximation are mathematically inconsistent, and therefore, rigorously, this decomposition formulation must be slightly modified by taking into account the correct B.C.s. In this thesis, we use the consistent boundary conditions to derive a slightly different decomposition formula which shows that the behavior of the diffracted wave at the edge is exactly just like an ordinary reflection—realizing the conjecture of Thomas Young in the 18th century. We also derived another decomposition formula which avoids mathematical discontinuity encountered by Rubinowicz. In the last section we demonstrate that our solution is consistent with that obtained by Sommerfeld in the rigorous 2-D plane-wave diffraction problem, so our formulation in this sense may describe more accurately the behavior of diffracted wave near the edge of the aperture than Kirchhoff's formula.

1 Introduction

1.1 Diffraction Integral Formulae

The diffraction of a scalar wave ψ satisfying the Helmholtz equation

$$(\nabla^2 + k^2) \psi = 0$$

has the solution (at the field point \vec{r}_f):

$$\psi(\vec{r}_f) = - \oint_{\mathcal{S}} \left(\psi \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n} \right) da \quad (1)$$

where the integral is performed on a closed surface \mathcal{S} which does not enclose any source of ψ , and \hat{n} is the outward normal of \mathcal{S} . Here, G is the Green's function satisfying

$$(\nabla^2 + k^2) G = -\delta^{(3)}(\vec{r} - \vec{r}_f).$$

For the problem of diffraction through an aperture on an infinite screen, one usually defines \mathcal{S} to be the union of the aperture, the screen, and the infinity. If the aperture is finite and the screen is opaque, one expects that ψ decreases fast to zero at infinity, and thus one only has to evaluate the integral on the aperture and the screen. Kirchhoff further assumed the boundary conditions

$$\begin{cases} \psi = \psi_s \text{ and } \frac{\partial \psi}{\partial n} = \frac{\partial \psi_s}{\partial n} & \text{on the aperture,} \\ \psi = 0 \text{ and } \frac{\partial \psi}{\partial n} = 0 & \text{on the screen,} \end{cases} \quad (2)$$

where ψ_s is the unperturbed source field. Also, he Kirchhoff the Green's function

$$G \equiv G_K \equiv \frac{1}{4\pi} \frac{e^{ik\|\vec{r}-\vec{r}_f\|}}{\|\vec{r}-\vec{r}_f\|}.$$

With these assumptions, Equation (1) can be reduced to

$$\psi(\vec{r}_f) = - \int_{\text{aperture}} \left(\psi_s \frac{\partial G_K}{\partial n} - G_K \frac{\partial \psi_s}{\partial n} \right) da. \quad (3)$$

However, the boundary conditions imposed by Kirchhoff is mathematically inconsistent, though it gives good approximations near the axis at far field zone. Sommerfeld, on the other hand, suggested another consistent boundary conditions

$$\begin{cases} \psi = \psi_s, & \text{on the aperture,} \\ \psi = 0, & \text{on the screen,} \end{cases} \quad (4)$$

and adopted Green's function of Dirichlet type which vanishes on the aperture and the screen. For example, for a planar screen,

$$G \equiv G_D \equiv G_K - G_K^* \equiv \frac{1}{4\pi} \frac{e^{ik\|\vec{r}-\vec{r}_f\|}}{\|\vec{r}-\vec{r}_f\|} - \frac{1}{4\pi} \frac{e^{ik\|\vec{r}-\vec{r}_f^*\|}}{\|\vec{r}-\vec{r}_f^*\|},$$

where \vec{r}_f^* is the mirror image of \vec{r}_f with respect to the screen. With these modifications, Equation (1) is reduced to

$$\psi(\vec{r}_f) = - \int_{\text{aperture}} \psi_s \frac{\partial G_D}{\partial n} da. \quad (5)$$

1.2 Maggi-Rubinowicz' Decomposition

With the Kirchhoff integral formula (Equation (3)), Rubinowicz was able to decompose the field $\psi(\vec{r}_f)$ into two parts [1]: one that appears only in the ordinary geometrical illuminated region is the source field evaluated at the field point \vec{r}_f ; the other one is a line integral along the edge of the aperture:

$$\psi(\vec{r}_f) = \begin{cases} \psi_s(\vec{r}_f) & , \vec{r}_f \in \text{illuminated region} \\ 0 & , \text{otherwise} \end{cases} - \frac{1}{4\pi} \oint_{\text{edge}} \psi_s(\vec{r}) \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} \right) \cdot d\vec{l}. \quad (6)$$

This formula applies to two special cases. The first one is the plane-wave-incidence case in which the source field is a plane wave: $\psi_s(\vec{r}) = e^{ik\rho}$, where ρ is the distance measured from \vec{r} to a constant phase plane of the incident field. If \vec{r} happens to lie on the edge of the aperture, then we denote ρ by ρ_s . Also, we denote $\hat{\rho}_s$ to be the unit vector in the direction of propagation of the incident field, as shown in Figure (1). The illuminated region is an oblique cylinder, as predicted by geometrical optics. Inside the line integral, $\rho_f \equiv \|\vec{r} - \vec{r}_f\|$ is the distance from \vec{r}_f to the edge \vec{r} , and $\hat{\rho}_f$ is the unit vector of $\vec{r} - \vec{r}_f$.

The second one is called the point-source-incidence case, in which the source field is a spherical wave: $\psi_s = e^{ik\rho}/\rho$, where $\rho \equiv \|\vec{r} - \vec{r}_s\|$ is the distance measured from \vec{r} to the position of the point source \vec{r}_s . As before, if \vec{r} happens to lie on the edge of the aperture, then we denote ρ by ρ_s , and use $\hat{\rho}_s$ to denote the unit vector of $\vec{r} - \vec{r}_s$. As expected, the illuminated region in this case is an oblique cone with vertex at \vec{r}_s , as shown in Figure (2).

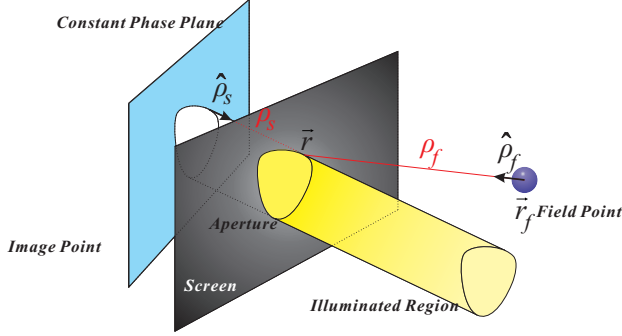


Figure (1)

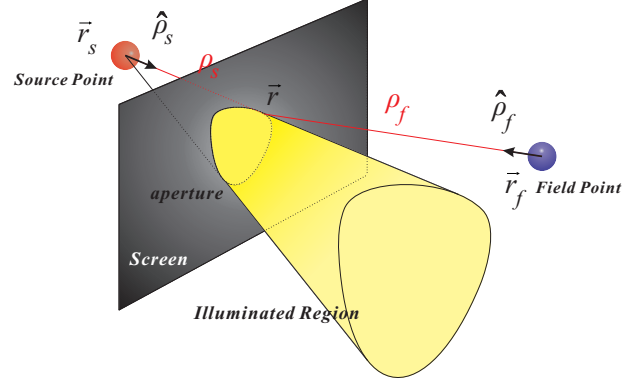


Figure (2)

This decomposition realized Young's interpretation for the diffraction phenomena: Young had once proposed that part of the incident field, which is called the reflected field, undergone a kind of *reflection* at the edge of the aperture, and the rest of the incident field, called geometrical field, just *went through* the aperture without any perturbation, and, the final diffraction wave was the interference of the two fields [2].

But as mentioned earlier, Equation (3) was derived based on inconsistent boundary conditions. Now, with the B.C.s proposed by Sommerfeld (Equation (4)), we are able to modify Equation (6) and obtain another similar expression which is not only mathematically-consistent, but also preserves Young's "field-splitting" interpretation.

Furthermore, Equation (6) is not entirely satisfactory in the sense that the two separated fields are not continuous at the boundary of the illuminated region. This problem had been discussed by John. S Asvestas, and he also gave an elegant "solid-angle" representation of ψ which successfully avoided such discontinuity. However, Asvestas' work was also based on Kirchhoff's formula, and therefore the mathematical inconsistency still exists. Besides that, the solid-angle representation derived by Asvestas does not reduce to electrostatic case in the long wavelength limit $k \rightarrow 0$. In this paper, we will give a slightly different decomposition formula with consistent boundary conditions and generalize the electrostatic result to the diffraction problem.

Another unsatisfactory feature of Rubinowicz' decomposition formula Equation (6) is that the functional form of the line integral is not what one would expect an ordinary reflection wave should be, due to some confusing factors in the integrand. And therefore in this paper, we will also

derive a neater representation of the line integral that mimics the behavior of ordinary reflection in geometrical optics.

In the last section, we'll explain why the boundary conditions (Equation (4)) is more suitable by compare the result of our work with Sommerfeld's 2-D straight-edge diffraction problem, which is the one of the few examples where the Helmholtz equation has an exact solution. We'll see that Kirchhoff's solution has a relative deviation from the exact solution.

2 Modified Expression for Rubinowicz' Decomposition Formula

With the boundary conditions Equation (4), we begin from Sommerfeld's integral formula Equation (5). Since $G_D (\partial\psi_s/\partial n) = 0$ on the aperture, we add it back into the integrand and rearrange a little:

$$\begin{aligned}\psi(\vec{r}_f) &= - \int_{\text{aperture}} \left(\psi_s \frac{\partial G_D}{\partial n} - G_D \frac{\partial \psi_s}{\partial n} \right) da \\ &= - \int_{\text{aperture}} \left(\psi_s \frac{\partial G_K}{\partial n} - G_K \frac{\partial \psi_s}{\partial n} \right) da + \int_{\text{aperture}} \left(\psi_s \frac{\partial G_K^*}{\partial n} - G_K^* \frac{\partial \psi_s}{\partial n} \right) da,\end{aligned}\tag{7}$$

and in the last line we identify the first integral is nothing but Equation (3) and thus equal to Equation (6). The second integral has exactly the same functional form as the first one, except for the replacement $\vec{r}_f \rightarrow \vec{r}_f^*$. So the final result is

$$\begin{aligned}\psi(\vec{r}_f) &= \begin{cases} \psi_s(\vec{r}_f) & , \vec{r}_f \in \text{illuminated region} \\ 0 & , \text{otherwise} \end{cases} \\ &\quad - \frac{1}{4\pi} \oint_{\text{edge}} \psi_s(\vec{r}) \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l}.\end{aligned}\tag{8}$$

(Note that \vec{r}_f^* is on the opposite side of the screen, so it always lies outside of the illuminated region, and thus there is no corresponding geometrical field.) Here $\hat{\rho}_f^*$ is the unit vector of $\vec{r} - \vec{r}_f^*$. The image term G_K^* contributes another line integral to the final expression, which now seems more ugly. Actually, Rubinowicz had also derived this result in his paper in 1917. However, for some reason he seemed to abandon this result and used Equation (6) in his successive papers. In the following sections, we'll show that, with some deformation, Equation (8) (or equivalently, Equation (5)) can take another form which has some merits mentioned in the introduction.

3 "Reflective" Representation

3.1 Motivations from Geometrical Optics

In ordinary geometrical optics, the reflection phenomena can be comprehended this way: given a source distribution, one draw the "image source" behind the "mirror," as shown in Figure (3), where there is a point real source labeled by S , and the reflected field is equal to the incident field from the image point S^* .

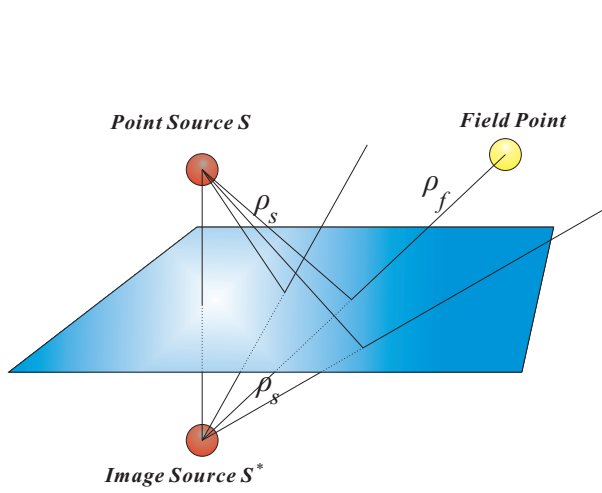


Figure (3)

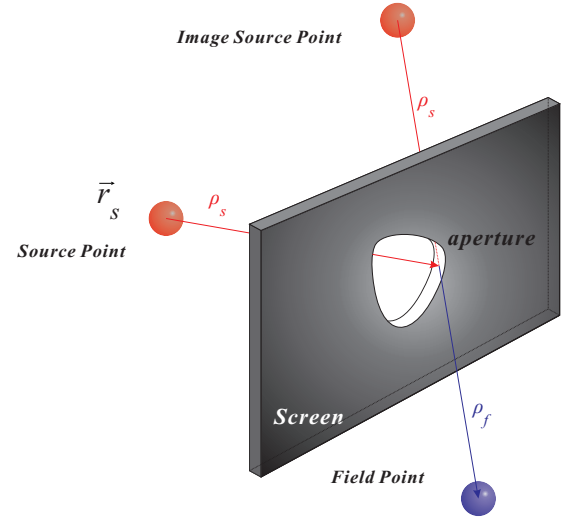


Figure (4)

So, if what Young really meant (in the early days when he saw the diffraction phenomena) by "reflection at the edge" was the reflection in geometrical optics, then we *expect* the line integral should take the form

$$\psi_{\text{reflection}} \sim \oint_{\text{edge}} e^{ik(\rho_s + \rho_f)} \text{ for plane wave,} \quad (9)$$

$$\psi_{\text{reflection}} \sim \oint_{\text{edge}} \frac{e^{ik(\rho_s + \rho_f)}}{\rho_s + \rho_f} \text{ for point source.} \quad (10)$$

Namely, as shown in Figure (4), we imagine that the screen has a finite thickness, and as the incident field ψ_s comes in, it is reflected by the "cut" around the aperture, and thus when the reflected field reaches the field point

\vec{r}_f , the incident field ψ_s has propagated for a total optical length $\rho_s + \rho_f$, and therefore $\psi_{\text{reflection}}$ should take the form as Equation (9) or (10).

These expectations actually can be accomplished by some deformations of Equation (8), but let's do it another way: to derive the $\psi_{\text{reflection}}$ from the beginning Equation (5), and this will make derivation more neater.

3.2 Reflection at the Boundary

Simplify Equation (5) a little, and we get

$$\begin{aligned}\psi(\vec{r}_f) &= - \int_{\text{aperture}} \psi_s \frac{\partial G_D}{\partial n} da = -2 \int_{\text{aperture}} \psi_s \frac{\partial G_K}{\partial n} da \\ &= \frac{1}{2\pi} \int_{\text{aperture}} \psi_s \frac{\partial}{\partial r} \left(\frac{e^{ikr}}{r} \right) \frac{\partial r}{\partial z} da\end{aligned}\quad (11)$$

where $r \equiv \|\vec{r} - \vec{r}_f\|$, and let \hat{e}_z be the inward unit normal to the screen ($\hat{e}_z = -\hat{n}$). To simplify the result, we consider the following two cases:

Case 1. *Plane Wave Diffraction*

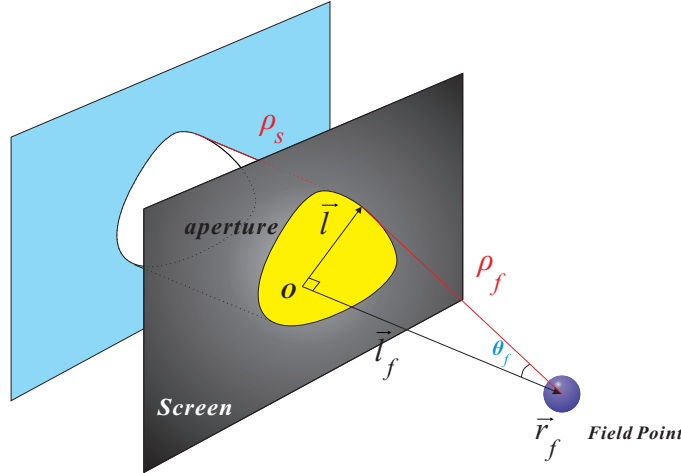


Figure (5)

We assume the wave propagates in the direction perpendicular to the screen—and it is reasonable to make this assumption since experimentally it is the most common configuration. Under this postulation, $\psi_s = e^{ik\rho_s}$ where ρ_s is now a constant quantity representing the distance from a constant phase plane to the screen. As shown in Figure (5), we make the projection of the field point \vec{r}_f on the screen, and denote it by O . Notice that O does not necessarily lie inside the aperture. Next, define two vectors \vec{l} and \vec{l}_f as

shown in Figure (5). Then every point along \vec{l} can be described by $s\vec{l}$, where $0 \leq s \leq 1$. Therefore

$$r \equiv \|\vec{r} - \vec{r}_f\| = \|s\vec{l} - \vec{l}_f\| = \sqrt{s^2 l^2 + l_f^2}$$

where l and l_f represent the magnitude of \vec{l} and \vec{l}_f , respectively. The area element on the aperture is

$$d\vec{a} = \vec{l} ds \times s d\vec{l},$$

and Equation (11) can be evaluated as

$$\begin{aligned} \psi(\vec{r}_f) &= -\frac{1}{2\pi} \int_{\text{aperture}} e^{ik\rho_s} \left(\frac{ik}{r} - \frac{1}{r^2} \right) e^{ikr} \frac{l_f}{r} da \\ &= -\frac{l_f e^{ik\rho_s}}{2\pi} \oint_{\text{edge}} \int_{s=0}^{s=1} \left(\frac{ik}{s^2 l^2 + l_f^2} - \frac{1}{(s^2 l^2 + l_f^2)^{3/2}} \right) e^{ik\sqrt{s^2 l^2 + l_f^2}} (\vec{l} ds \times s d\vec{l}) \cdot \hat{e}_z. \\ &= -\frac{l_f e^{ik\rho_s}}{2\pi} \oint_{\text{edge}} \frac{(\vec{l} \times d\vec{l}) \cdot \hat{e}_z}{l^2} \left(\frac{e^{ik\sqrt{s^2 l^2 + l_f^2}}}{\sqrt{s^2 l^2 + l_f^2}} \right) \Big|_{s=0}^{s=1} \end{aligned}$$

But

$$(\vec{l} \times d\vec{l}) \cdot \hat{e}_z = l^2 d\phi,$$

where ϕ is the angle subtended by the arc of the boundary of the aperture as measured from O . So

$$\psi(\vec{r}_f) = \frac{l_f e^{ik\rho_s}}{2\pi} \oint_{\text{edge}} d\phi \left(\frac{e^{ikl_f}}{l_f} - \frac{e^{ik\sqrt{l^2 + l_f^2}}}{\sqrt{l^2 + l_f^2}} \right) = \frac{1}{2\pi} \oint_{\text{edge}} d\phi \left(\psi_s(\vec{r}_f) - e^{ik(\rho_s + \rho_f)} \cos \theta_f \right)$$

where θ_f is the angle indicated in Figure (5).

To separate the geometrical and reflected fields apart, we perform the line integral to the first term of the integrand:

$$\frac{1}{2\pi} \oint_{\text{edge}} \psi_s(\vec{r}_f) d\phi = \begin{cases} \psi_s(\vec{r}_f) & , \text{if } O \text{ lies inside the aperture} \\ 0 & , \text{otherwise} \end{cases}.$$

Since O lies inside the aperture if and only if $\vec{r}_f \in \text{illuminated region}$, so

$$\psi(\vec{r}_f) = \begin{cases} \psi_s(\vec{r}_f) & , \vec{r}_f \in \text{illuminated region} \\ 0 & , \text{otherwise} \end{cases} - \frac{1}{2\pi} \oint_{\text{edge}} e^{ik(\rho_s + \rho_f)} \cos \theta_f d\phi \quad (12)$$

Case 2. Point Source Diffraction

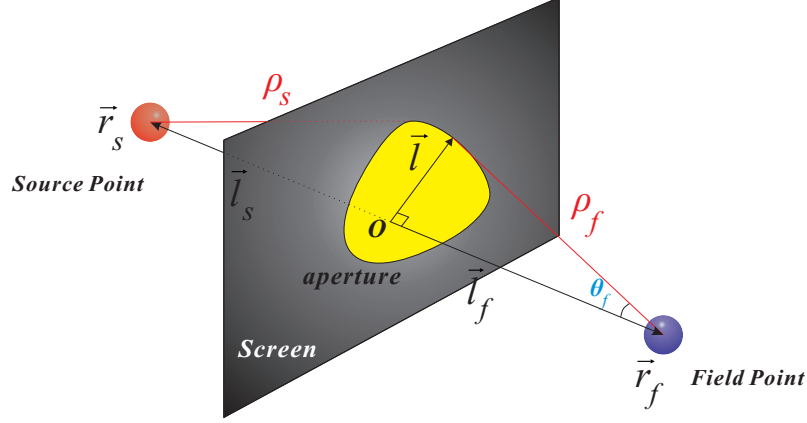


Figure (6)

Similar to the previous case, we attempt to assume that $\vec{r}_f - \vec{r}_s$ is perpendicular to the screen, that is, we confine \vec{r}_f to lie on the central axis as shown in Figure (6). But this time the assumption is totally unreasonable—after all we cannot restrict the position of \vec{r}_f . We believe that this formulation can be generalized without this assumption, but for the present we'll consider this special case, and try to elucidate the idea of "reflection at the boundary."

Identify $\psi_s = e^{ik\rho}/\rho$, where $\rho \equiv \|\vec{r} - \vec{r}_s\|$, and define O , \vec{l}_f , \vec{l} as before. Here, we define a new vector \vec{l}_s , to be the vector from O to \vec{r}_s . Therefore

$$\rho = \|\vec{s}\vec{l} - \vec{l}_s\| = \sqrt{s^2l^2 + l_s^2},$$

and Equation.(11) can be evaluated as

$$\begin{aligned} \psi(\vec{r}_f) &= -\frac{1}{2\pi} \int_{\text{aperture}} \frac{e^{ik\rho}}{\rho} \left(\frac{ik}{r} - \frac{1}{r^2} \right) e^{ikr} \frac{l_f}{r} da \\ &= -\frac{l_f}{2\pi} \oint_{\text{edge}} \int_{s=0}^{s=1} \frac{e^{ik(\sqrt{s^2l^2+l_f^2} + \sqrt{s^2l^2+l_s^2})}}{\sqrt{s^2l^2+l_s^2}} \left(\frac{ik}{s^2l^2+l_f^2} - \frac{1}{(s^2l^2+l_f^2)^{3/2}} \right) (\vec{l}ds \times s d\vec{l}) \cdot \hat{e}_z \\ &= -\frac{l_f}{2\pi} \oint_{\text{edge}} \frac{(\vec{l} \times d\vec{l}) \cdot \hat{e}_z}{l^2} \left(\frac{e^{ik\sqrt{s^2l^2+l_f^2}}}{\sqrt{s^2l^2+l_f^2}} \frac{e^{ik\sqrt{s^2l^2+l_s^2}}}{\sqrt{s^2l^2+l_f^2} + \sqrt{s^2l^2+l_s^2}} \right) \Big|_{s=0}^{s=1} \end{aligned}$$

So

$$\psi(\vec{r}_f) = \frac{1}{2\pi} \oint_{\text{edge}} \left(\frac{e^{ik(l_f+l_s)}}{l_f+l_s} - \frac{e^{ik(\rho_s+\rho_f)}}{\rho_s+\rho_f} \frac{l_f}{\rho_f} \right) d\phi = \frac{1}{2\pi} \oint_{\text{edge}} \left(\psi_s(\vec{r}_f) - \frac{e^{ik(\rho_s+\rho_f)}}{\rho_s+\rho_f} \cos \theta_f \right) d\phi$$

where θ_f has the same definition as before.

Again the geometrical field can be separated out by the same method, and we get the final result:

$$\psi(\vec{r}_f) = \begin{cases} \psi_s(\vec{r}_f) & , \vec{r}_f \in \text{illuminated region} \\ 0 & , \text{otherwise} \end{cases} - \frac{1}{2\pi} \oint_{\text{edge}} \frac{e^{ik(\rho_s + \rho_f)}}{\rho_s + \rho_f} \cos \theta_f d\phi \quad (13)$$

4 Solid-Angle Representation

4.1 Motivation from Electrostatics

Although Equation (8) is a mathematically-consistent solution, it still exhibits the same problem as what Rubinowicz encountered in his solution: the geometrical and reflected fields are discontinuous at the boundary of the illuminated region. To overcome this problem, we seek for the analogy in electrostatics: Consider a grounded infinite conducting plane with a finite insulating region σ at which the potential is held at a constant value V_0 :

$$V = \begin{cases} V_0 & , \text{on the insulating region } \sigma \\ 0 & , \text{on the conducting plane} \end{cases}.$$

Assume there is no other source charge in the half space $z > 0$, and therefore $\nabla^2 V = 0$ there. This boundary value problem has the solution

$$V(\vec{r}_f) = \frac{\Omega_f}{2\pi} V_0 \quad (14)$$

where Ω_f is the solid angle subtended by the region σ as observed at the field point \vec{r}_f .

Inspired by the electrostatic result, we attempt a solution for diffraction problem of the form

$$\psi(\vec{r}_f) = \frac{\Omega_f}{2\pi} \psi_s(\vec{r}_f) + (\text{a line-integral}),$$

namely, we expect the geometrical field to take the similar form of Equation (14), while the reflected field remains a line-integral around the boundary of the aperture. The advantage of this formulation is that both the geometrical and reflected fields now vary continuously, without any jump discontinuity across the boundary of the illuminated region. Different from Asvestas' work, in this formula, we see that as the field point \vec{r}_f approaches to the aperture, then $\Omega_f \rightarrow 2\pi$, and the geometrical field $\rightarrow \psi_s$ while we expect the reflected field to vanish totally. That is, if we reside on the aperture, we are exposing ourself to the source ψ_s without the influence of the edge.

4.2 Inverse Cone

We begin from Equation (7), and define

$$J \equiv - \oint_{\text{aperture}} \left(\psi_s \frac{\partial G_K}{\partial n} - G_K \frac{\partial \psi_s}{\partial n} \right) da,$$

$$J^* \equiv \oint_{\text{aperture}} \left(\psi_s \frac{\partial G_K^*}{\partial n} - G_K^* \frac{\partial \psi_s}{\partial n} \right) da.$$

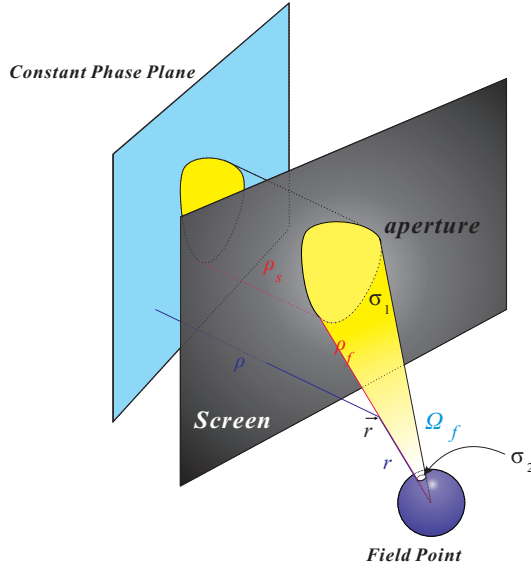


Figure (7)

To evaluate J , we do a trick slightly different from what Rubinowicz did. As shown in Figure (7), we make an auxiliary surface with the vertex at the *field point*, and make a small ball centered at the field point. Define σ_1 to be the surface of the cone outside the small ball while σ_2 to be the surface of the small ball inside the cone. Apply the divergence theorem to the region enclosed by σ_1 , σ_2 and the aperture:

$$\int_{\text{aperture}} \vec{F} \cdot d\vec{a} + \int_{\sigma_1} \vec{F} \cdot d\vec{a} + \int_{\sigma_2} \vec{F} \cdot d\vec{a} = 0$$

where $\vec{F} \equiv \psi_s \vec{\nabla} G_K - G_K \vec{\nabla} \psi_s$, and $d\vec{a}$ is the vectorial area element pointing outwardly from the volume enclosed. If we let the radius of the small ball approach to zero, then

$$\int_{\sigma_2} \vec{F} \cdot d\vec{a} \rightarrow -\frac{\Omega_f}{4\pi} \left(\vec{\nabla} \cdot \vec{F} \right) = -\frac{\Omega_f}{4\pi} \psi_s (\vec{r}_f),$$

where, as desired, Ω_f is the solid angle subtended by the aperture as observed at the field point \vec{r}_f . So

$$J = \frac{\Omega_f}{4\pi} \psi_s(\vec{r}_f) - \int_{\sigma_1} \vec{F} \cdot d\vec{a}.$$

The surface integral can be evaluated by the same trick presented by Rubinowicz, as discuss in the following two cases:

Case 1. Plane Wave Diffraction

On the auxiliary surface, $\vec{\nabla} G_K = 0$, and therefore

$$\begin{aligned} \int_{\sigma_1} \vec{F} \cdot d\vec{a} &= \int_{\sigma_1} \frac{e^{ikr}}{r} ik e^{ik\rho} \hat{\rho} \cdot \left(\frac{\hat{\rho}_f \times d\vec{l}}{\rho_f} r dr \right) = \int_{\sigma_1} ik e^{ik(r+\rho)} \hat{\rho}_s \cdot \left(\frac{\hat{\rho}_f \times d\vec{l}}{\rho_f} dr \right) \\ &= \oint_{\text{edge}} ik \hat{\rho}_s \cdot \left(\frac{\hat{\rho}_f \times d\vec{l}}{\rho_f} \right) \int_{r=0}^{r=\rho_f} e^{ik(r+\rho)} dr. \end{aligned}$$

From Figure (7), we have the relation

$$\rho = -(\rho_f - r) \hat{\rho}_s \cdot \hat{\rho}_f + \rho_s$$

and thus

$$\int_{r=0}^{r=\rho_f} e^{ik(r+\rho)} dr = e^{ik(-\rho_f \hat{\rho}_s \cdot \hat{\rho}_f + \rho_s)} \int_{r=0}^{r=\rho_f} e^{ikr(1+\hat{\rho}_s \cdot \hat{\rho}_f)} dr = \frac{1}{ik} \frac{e^{ik(\rho_s + \rho_f)} - e^{ik(-\rho_f \hat{\rho}_s \cdot \hat{\rho}_f + \rho_s)}}{1 + \hat{\rho}_s \cdot \hat{\rho}_f}$$

and

$$\int_{\sigma_1} \vec{F} \cdot d\vec{a} = \oint_{\text{edge}} e^{ik\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(1 - e^{-ik\rho_f(1+\hat{\rho}_s \cdot \hat{\rho}_f)} \right) \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} \right) \cdot d\vec{l}.$$

Therefore

$$J = \frac{\Omega_f}{4\pi} \psi_s(\vec{r}_f) - \frac{1}{4\pi} \oint_{\text{edge}} e^{ik\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(1 - e^{-ik\rho_f(1+\hat{\rho}_s \cdot \hat{\rho}_f)} \right) \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} \right) \cdot d\vec{l}.$$

To evaluate J^* , we use the result from Section 2:

$$J^* = \frac{1}{4\pi} \oint_{\text{edge}} e^{ik\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l}.$$

Finally, we combine J and J^* :

$$\begin{aligned} \psi(\vec{r}_f) &= \frac{\Omega_f}{4\pi} \psi_s(\vec{r}_f) \\ &\quad - \frac{1}{4\pi} \oint_{\text{edge}} e^{ik\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left[\left(1 - e^{-ik\rho_f(1+\hat{\rho}_s \cdot \hat{\rho}_f)} \right) \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} \right) - \left(\frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \right] \cdot d\vec{l}. \end{aligned} \tag{15}$$

Although this result fits our demand— $\psi(\vec{r}_f)$ is now expressed in terms of the solid angle Ω_f —it is still unsatisfactory since the denominator of the geometrical part is 4π instead of 2π . Accordingly, if \vec{r}_f approaches to the aperture, the geometrical part only gives us one half of the total source wave ψ_s , and thus the reflected part must contribute the rest half part. To fix the problem, we take the long wavelength limit $k \rightarrow 0$, and thus

$$\psi_s \rightarrow e^{i0\rho} = 1,$$

and Equation (15) must be identical to Equation (14):

$$\psi(\vec{r}_f) \rightarrow \frac{\Omega_f}{4\pi} + \frac{1}{4\pi} \oint_{\text{edge}} \frac{1}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l} \equiv \frac{\Omega_f}{2\pi}.$$

So we have a line-integral representation of solid angle:

$$\frac{\Omega_f}{4\pi} = \frac{1}{4\pi} \oint_{\text{edge}} \frac{1}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l}. \quad (16)$$

This equation has also been derived by Asvestas [3], and by Yih-Yuh Chen [4] from a more elegant perspective. Note that since the wave number k vanishes,

the vector $\hat{\rho}_s$ now can point in an arbitrary direction, so the representation above is *not unique*.

Finally, we construct the desired $\psi(\vec{r}_f)$ by adding Equation (16) into Equation (15)

$$\psi(\vec{r}_f) = \frac{\Omega_f}{2\pi} \psi_s(\vec{r}_f) - \frac{1}{4\pi} \oint_{\text{edge}} \psi_s \frac{e^{ik\rho_f}}{\rho_f} \left[\begin{array}{l} \left(1 - e^{-ik\rho_f(1+\hat{\rho}_s \cdot \hat{\rho}_f)} \right) \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} \right) \\ - \left(1 - e^{-ik(\rho_s+\rho_f)} \right) \left(\frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \end{array} \right] \cdot d\vec{l}.$$

Case 2. Point Source Diffraction

Again, on the auxiliary surface, $\vec{\nabla}G_K = 0$, and

$$\begin{aligned} \int_{\sigma_1} \vec{F} \cdot d\vec{a} &= \int_{\sigma_1} \frac{e^{ik(r+\rho)}}{r} \left(\frac{ik}{\rho} - \frac{1}{\rho^2} \right) \hat{\rho} \cdot \left(\frac{\hat{\rho}_f \times d\vec{l}}{\rho_f} r dr \right) = \int_{\sigma_1} e^{ik(r+\rho)} \left(\frac{ik}{\rho} - \frac{1}{\rho^2} \right) \frac{\vec{\rho}_s}{\rho} \cdot \left(\frac{\hat{\rho}_f \times d\vec{l}}{\rho_f} dr \right) \\ &= \oint_{\text{edge}} \vec{\rho}_s \cdot \left(\frac{\hat{\rho}_f \times d\vec{l}}{\rho_f} \right) \int_{r=0}^{r=\rho_f} e^{ik(r+\rho)} \left(\frac{ik}{\rho^2} - \frac{1}{\rho^3} \right) dr \end{aligned}$$

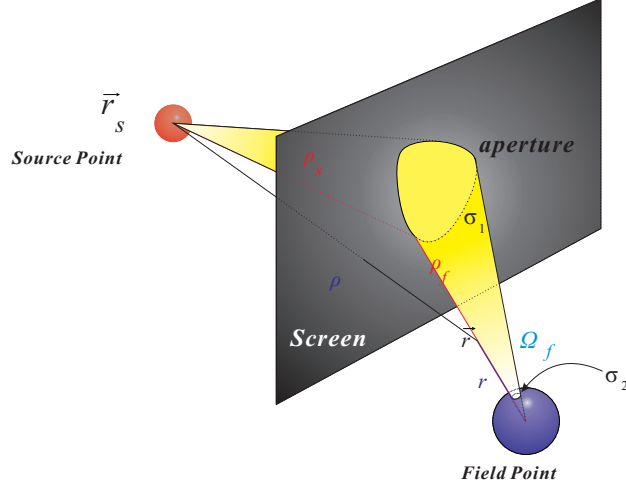


Figure (8)

From Figure (8), we have the relation

$$\rho^2 = \rho_s^2 + (\rho_f - r)^2 - 2\rho_s(\rho_f - r)\hat{\rho}_s \cdot \hat{\rho}_f.$$

Differentiate it with respect to r :

$$\rho \left(1 + \frac{d\rho}{dr} \right) = r + \rho - \rho_f + \rho_s \hat{\rho}_s \cdot \hat{\rho}_f,$$

and (follow Rubinowicz' calculation)

$$\int_{r=0}^{r=\rho_f} e^{ik(r+\rho)} \left(\frac{ik}{\rho^2} - \frac{1}{\rho^3} \right) dr = \frac{e^{ik(r+\rho)}}{\rho(r+\rho-\rho_f+\rho_s\hat{\rho}_s \cdot \hat{\rho}_f)} \Big|_{r=0}^{r=\rho_f} = \frac{e^{ik(\rho_f+\rho_s)}}{\rho_s^2(1+\hat{\rho}_s \cdot \hat{\rho}_f)} - \frac{e^{ik\rho_0}}{\rho_0^2(1+\hat{\rho}_0 \cdot \hat{\rho}_f)}$$

where $\vec{\rho}_0 \equiv \vec{\rho}_s - \vec{\rho}_f$ is the vector from \vec{r}_s to \vec{r}_f . So

$$\int_{\sigma_1} \vec{F} \cdot d\vec{a} = \oint_{\text{edge}} \frac{e^{ik\rho_s}}{\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{1}{1+\hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\rho_s^2}{\rho_0^2} \frac{e^{-ik(\rho_s+\rho_f-\rho_0)}}{1+\hat{\rho}_0 \cdot \hat{\rho}_f} \right) (\hat{\rho}_s \times \hat{\rho}_f) \cdot d\vec{l}.$$

Therefore

$$J = \frac{\Omega_f}{4\pi} \psi_s(\vec{r}_f) - \frac{1}{4\pi} \oint_{\text{edge}} \frac{e^{ik\rho_s}}{\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{1}{1+\hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\rho_s^2}{\rho_0^2} \frac{e^{-ik(\rho_s+\rho_f-\rho_0)}}{1+\hat{\rho}_0 \cdot \hat{\rho}_f} \right) (\hat{\rho}_s \times \hat{\rho}_f) \cdot d\vec{l}.$$

Again, we use the result from Section 2:

$$J^* = \frac{1}{4\pi} \oint_{\text{edge}} \frac{e^{ik\rho_s}}{\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1+\hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l}.$$

Combine J and J^* , we have

$$\psi(\vec{r}_f) = \frac{\Omega_f}{4\pi} \psi_s(\vec{r}_f) - \frac{1}{4\pi} \oint_{\text{edge}} \frac{e^{ik\rho_s}}{\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left[\begin{aligned} &\left(\frac{1}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\rho_s^2}{\rho_0^2} \frac{e^{-ik(\rho_s + \rho_f - \rho_0)}}{1 + \hat{\rho}_0 \cdot \hat{\rho}_f} \right) (\hat{\rho}_s \times \hat{\rho}_f) \\ &- \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \end{aligned} \right] \cdot d\vec{l}. \quad (17)$$

To construct the correct factor $1/2\pi$, we use Equation (16) to add another $(\Omega_f/4\pi) \psi_s$ to the geometric wave. But note that $\hat{\rho}_s$ in Equation (16) is an arbitrary *constant* vector, and in Equation (17) $\hat{\rho}_s$ represents a varying vector that changes its direction as we integrate along the edge. Thus we must specify one direction for $\hat{\rho}_s$ in Equation (16) so that we can insert it into Equation (17). The result is most symmetric if we adopt $\hat{\rho}_s \equiv \hat{\rho}_0$ in Equation (16)

$$\psi(\vec{r}_f) = \frac{\Omega_f}{2\pi} \psi_s(\vec{r}_f) - \frac{1}{4\pi} \oint_{\text{edge}} \psi_s \frac{e^{ik\rho_f}}{\rho_f} \left[\begin{aligned} &\left(\frac{1}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\rho_s^2}{\rho_0^2} \frac{e^{-ik(\rho_s + \rho_f - \rho_0)}}{1 + \hat{\rho}_0 \cdot \hat{\rho}_f} \right) (\hat{\rho}_s \times \hat{\rho}_f) \\ &- \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} + e^{-ik\rho_f} \left(\frac{\hat{\rho}_0 \times \hat{\rho}_f^*}{1 + \hat{\rho}_0 \cdot \hat{\rho}_f^*} \right) \end{aligned} \right] \cdot d\vec{l}.$$

We have seen that in both cases the field $\psi(\vec{r}_f)$ has the form

$$\psi(\vec{r}_f) = \frac{\Omega_f}{2\pi} \psi_s(\vec{r}_f) - \frac{1}{4\pi} \oint_{\text{edge}} \psi_s \frac{e^{ik\rho_f}}{\rho_f} [\dots] \cdot d\vec{l},$$

and it is obvious that both the geometrical and the line integral parts of ψ are now continuous across the surface of illuminated region. However, the integrand in $[\dots]$ now depends on the type of the source. There is another point to be mentioned: since $\Omega_f \propto 1/\rho_f^2$, in the far zone the geometrical field is overwhelmed by the reflected field, which is proportional to $1/\rho_f$ of the source wave. The situation is reversed in the near zone, of course.

5 Comparison of Boundary Conditions

As discussed in previous sections, the boundary conditions based on Kirchhoff's theory is mathematically inconsistent, and by using the proper Green's function, the diffraction theory can be transformed into a boundary value

problem of Dirichlet type which is mathematically admissible. However, this is not the whole story. As the source wave ψ_s propagates toward the aperture, the wave must be modified by the presence of the opaque screen, and thus ψ is not exactly equal to ψ_s , the unperturbed source, on the aperture. So the boundary values Equation (4) imposed earlier is still, unsatisfactory in the physical sense.

However, Sommerfeld has solved a $2-D$ straight edge diffraction problem rigorously without using the unperturbed source wave as boundary values [5], and we'll see in a moment that, by some deformation of Equation (8), the functional form of our solution is very closed to that of Sommerfeld's, and, therefore, we may regard Equation (4) as an acceptable approximation to the real, rigorous solution.

5.1 Approximated Solution for a Point Source

Consider an infinite half plane lying on $z = 0$ and $x > 0$, with a point source lying in the region $z < 0$ as before. The solution ψ in the space $z > 0$ can be solved by Equation (8)

$$\psi(\vec{r}_f) = \begin{cases} \psi_s(\vec{r}_f) & , \vec{r}_f \in \text{illuminated region} \\ 0 & , \text{otherwise} \end{cases} - \frac{1}{4\pi} \int_{\text{edge}} A \frac{e^{ik\rho_s}}{\rho_s} \frac{e^{ik\rho_f}}{\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l}.$$

where the line integral is performed along the infinite straight edge¹. The amplitude A of the source field is now expressed explicitly for later convenience.

In the far field region $k\rho_f \gg 1$, we apply stationary-phase approximation to evaluate the reflected field:

$$I = \frac{1}{4\pi} \int_{\text{edge}} A \frac{e^{ik(\rho_s + \rho_f)}}{\rho_s \rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot d\vec{l}. \quad (18)$$

The stationary point occurs when $\nabla(\rho_s + \rho_f) \cdot d\vec{l} = 0$, and we expand the phase at the stationary point:

$$\begin{aligned} \rho_s + \rho_f &= (\rho_s + \rho_f) \Big|_0 + \nabla(\rho_s + \rho_f) \Big|_0 \cdot \delta\vec{l} + \frac{1}{2} \delta\vec{l} \cdot \left(\frac{\hat{\mathbf{1}} - \hat{\rho}_s \hat{\rho}_s}{\rho_s} + \frac{\hat{\mathbf{1}} - \hat{\rho}_f \hat{\rho}_f}{\rho_f} \right) \Big|_0 \cdot \delta\vec{l} \\ &= (\rho_s + \rho_f) \Big|_0 + \frac{1}{2} \frac{\rho_s + \rho_f}{\rho_s \rho_f} \delta\vec{l}^2 \sin^2 \left(d\vec{l}, \hat{\rho}_s \right) \Big|_0, \end{aligned} \quad (19)$$

¹And thus the integral sign \oint is used instead of \oint .

where the subscript 0 denotes the stationary point, which in this special case is the point on the edge nearest to \vec{r}_f ; also, $\hat{\mathbf{1}}$ is the identity operator in three dimensional space, and $\sin \left(d\vec{l}, \hat{\rho}_s \right)$ is the sine of the angle between $d\vec{l}$ and $\hat{\rho}_s$. Insert Equation (19) into Equation (18), and perform the Gaussian integral, we get

$$I \simeq \frac{1}{4\pi} \frac{Ae^{ik(\rho_s+\rho_f)}}{\rho_s\rho_f} \left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot \frac{d\vec{l}}{\|d\vec{l}\|} \Big|_0 \sqrt{\frac{2\pi i}{k}} \frac{\rho_s\rho_f}{\rho_s + \rho_f} \frac{1}{\sin \left(d\vec{l}, \hat{\rho}_s \right)} \Big|_0.$$

But

$$\left(\frac{\hat{\rho}_s \times \hat{\rho}_f}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_s \times \hat{\rho}_f^*}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \cdot \frac{d\vec{l}}{\|d\vec{l}\|} \Big|_0 = \left(\frac{\hat{\rho}_f \cdot \hat{n}}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_f^* \cdot \hat{n}}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} \right) \sin \left(d\vec{l}, \hat{\rho}_s \right) \Big|_0.$$

where

$$\hat{n} = \frac{d\vec{l} \times \hat{\rho}_s}{\|d\vec{l} \times \hat{\rho}_s\|}$$

is the unit outward normal of the geometric light cone. To simplify the factor in the parenthesis, refer to Figure (9), we have

$$\frac{\hat{\rho}_f \cdot \hat{n}}{1 + \hat{\rho}_s \cdot \hat{\rho}_f} - \frac{\hat{\rho}_f^* \cdot \hat{n}}{1 + \hat{\rho}_s \cdot \hat{\rho}_f^*} = \frac{-\sin(\phi - \alpha)}{1 + \cos(\phi - \alpha)} - \frac{\sin(\phi + \alpha)}{1 + \cos(\phi + \alpha)} = -\frac{2 \sin \phi}{\cos \alpha + \cos \phi}$$

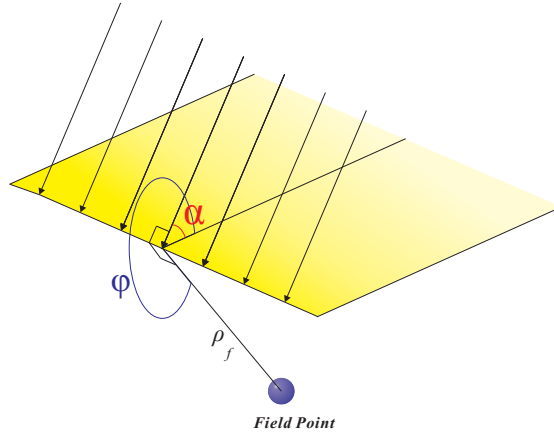


Figure (9)

So finally,

$$I \simeq -A \frac{e^{ik(\rho_s + \rho_f)}}{\sqrt{2\pi k \rho_s \rho_f}} \frac{2 \sin \phi}{\cos \alpha + \cos \phi} \frac{e^{i\frac{\pi}{4}}}{\sqrt{\rho_s + \rho_f}} \quad (20)$$

The factor $e^{i\frac{\pi}{4}}$ can explain the reason why the diffraction pattern in the water has a phase delay compared to the incident wave.

5.2 Approximated Solution for Plane Waves

In the case of plane-wave-incidence, we can obtain the solution from Equation (20) by taking the limit

$$\rho_s \rightarrow \infty, A \rightarrow \infty, \text{ while keeping } \frac{A}{\rho_s} \rightarrow \text{finite number, taken to be 1}$$

The result is

$$I \rightarrow -\frac{e^{ik(\rho_s + \rho_f)}}{\sqrt{2\pi k \rho_f}} \left(\frac{2 \sin \phi}{\cos \alpha + \cos \phi} \right) e^{i\frac{\pi}{4}} = -\frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \left(\frac{1}{\cos \frac{\phi-\alpha}{2}} - \frac{1}{\cos \frac{\phi+\alpha}{2}} \right) \frac{\cos \frac{\phi}{2}}{\sin \frac{\alpha}{2}}.$$

Here $e^{ik\rho_s}$ has been dropped since Sommerfeld assumed that the plane wave has phase 0 right at $\rho_f = 0$.

So the total field is

$$\psi(\vec{r}_f) \simeq \begin{cases} \psi_s(\vec{r}_f) + \frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \left(\frac{1}{\cos \frac{\phi-\alpha}{2}} - \frac{1}{\cos \frac{\phi+\alpha}{2}} \right) \frac{\cos \frac{\phi}{2}}{\sin \frac{\alpha}{2}}, & \vec{r}_f \in \text{illuminated region} \\ \frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \left(\frac{1}{\cos \frac{\phi-\alpha}{2}} - \frac{1}{\cos \frac{\phi+\alpha}{2}} \right) \frac{\cos \frac{\phi}{2}}{\sin \frac{\alpha}{2}}, & \text{otherwise} \end{cases}.$$

In comparison with Sommerfeld's solution [5]

$$\psi(\vec{r}_f) \simeq \begin{cases} \psi_s(\vec{r}_f) + \frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \left(\frac{1}{\cos \frac{\phi-\alpha}{2}} - \frac{1}{\cos \frac{\phi+\alpha}{2}} \right), & \vec{r}_f \in \text{illuminated region} \\ \frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \left(\frac{1}{\cos \frac{\phi-\alpha}{2}} - \frac{1}{\cos \frac{\phi+\alpha}{2}} \right), & \text{otherwise} \end{cases}.$$

we see that, apart from the factor $\frac{\cos \frac{\phi}{2}}{\sin \frac{\alpha}{2}}$, the two representations are similar.

The discrepancy results from the different boundary conditions as discussed before.

If, however, we use the Kirchhoff's integral formula (with mathematically inconsistent B.C.s), we would obtain

$$\psi(\vec{r}_f) \simeq \begin{cases} \psi_s(\vec{r}_f) + \frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \tan\left(\frac{\phi-\alpha}{2}\right) & , \vec{r}_f \in \text{illuminated region} \\ \frac{1+i}{4\sqrt{\pi k \rho_f}} e^{ik\rho_f} \tan\left(\frac{\phi-\alpha}{2}\right) & , \text{otherwise} \end{cases},$$

which has a different functional form from Sommerfeld's solution.

6 Conclusion

By using mathematically consistent boundary conditions, we have seen that Rubinowicz' decomposition formulation can be more useful: the functional form of the line integral becomes much neater and admits a simple interpretation of *reflection at edges*. The formulation also provides us a different approach that makes the diffraction phenomena similar to the electrostatic problem by using solid angle representation for the geometrical field. Finally, the diffracted field predicted by this formulation is much closer to the physical solution, as discussed in the last section.

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